ABSTRACT. The aim of this paper is to analyse the functions of semiotic mediation in a long term teaching experiment on the plane representation of three-dimensional space by means of perspective drawing, that has been tried out from grade 2 to grade 5 in three different classrooms within the research project Mathematical Discussion. On the one side, drawing has a functional role in the overall development of the child; on the other side, perspective drawing has a phenomenological role in the genesis of modern geometry. The experiment aims at connecting (1) pupils’ spatial experiences to the development of the geometry of three-dimensional space and (2) pupils’ drawing experiences to the geometry of two-dimensional space, up to the mastery of early geometrical strategies of plane representation of space. Classroom activity alternates individual problems and classroom discussions orchestrated by the teacher. The paper is divided into several parts: after a brief introduction containing some contextual information (§ 1, 2), the problem of the social construction of knowledge is addressed and some theoretical constructs mainly borrowed from the Vygotskian school are elaborated (§ 3); then two analyses of the experiment are made, according to the motives of activity (§ 4) and to the sequence of actions (§ 5); finally the role of semiotic mediation in the whole experiment is analysed (§ 6); in the final section (§ 7) some results are recapitulated and compared with the literature on the teaching and learning of geometry and the function of semiotic mediation is discussed with reference to the other distinctive features of the teaching experiment.

1. A PROTOCOL

What did I understand? Perspective is a solvable problem.

[...] [1]Piero della Francesca wished to explain how to draw a squared floor in any room.

[2]This ideal room has a perfect floor for the oblique lines only: horizontal lines have no measure, i.e. Piero della Francesca has not given any explanation how to “degrade” distances.

[3]Concerning the above “room”, my eyes are on the wall in a frontal and central position, because I have drawn the room as Costanza told when we discussed perspective.

To draw suitably, it is necessary to respect the rules of “drawing” and some others that are not in Piero’s discourse.

Now we have to be able to draw on a plane whatever one wishes to draw, with proportions and similar shapes: so the shift from three to two-dimensional drawing is exact.

When we have to draw a table or a flat object, to be represented exactly from a frontal perspective, the two lines which “surround” the “figure” are drawn as follows: we imagine the two lines as converging in a point.

I have forgotten an important thing:

The point does not exist, but is imagined by geometers. This point is linked with others, but only in the imagination. Actually it is an intersection point [...].

(Elisa, 4th grade: excerpt from the written individual comment on the collective reading of two excerpts from De Prospectiva Pingendi, by Piero della Francesca; emphasis is in the text; the numbers have been added for the sake of further reference.)

2. INTRODUCTION

The above protocol is taken from the classroom data of a long term teaching experiment concerning the representation of three-dimensional space by means of perspective drawing (from now on it will be referred to as the teaching experiment on perspective drawing). The experiment aimed at connecting pupils’ spatial experiences to the development of three-dimensional geometry and pupils’ drawing experiences to the development of two dimensional geometry up to the mastery of early geometrical strategies of plane representation of space.
However, the quoted protocol suggests that other phenomena have happened: for instance Elisa is well aware of and can control her own intellectual processes [7]; she is able to distance herself from her products [2, 3, 4]; she can locate herself historically with respect to her own history, classroom history [5] and even human history [1, 5]; she knows the value of a general method that applies to representing any object [5, 6]; she is aware of the epistemological complexity of relationships between mathematics and reality [8]. Among other things these are symptoms that metacognitive activity is taking place, in spite of the young age of Elisa. If Elisa is not a special child but is representative of most of her schoolfellows (as she is), what is the reason for this unexpected result? Perhaps it could be found in the quality of mathematical activity she has experienced from the beginning of primary school.

The aim of this paper is to analyse the long term teaching experiment, in which Elisa took part, by means of the theoretical construct of semiotic mediation (Vygotsky, 1978) in an attempt to substantiate its crucial effect on pupils’ learning and metalearning.

Before entering the core of the paper, some contextual information about the teaching experiment is needed. The experiment is the continuation of another teaching experiment on the coordination of spatial perspectives in the 1st and 2nd grades (Bartolini Bussi, in press). The teaching experiment unfolded over a period of three years as a part of the project Mathematical Discussion (Bartolini Bussi, 1991). It was carried out almost completely in three classrooms from grade 2 to 5 and alternated individual and collective tasks: examples of the former are the production of real-life drawings and of written essays and the solution of related geometrical problems; examples of the latter are the readings of historical sources and the discussions orchestrated by the teacher.

Both experiments are conceived as cases of research for innovation (Bartolini Bussi, 1994a) as they aim at turning into reality some anticipated events that are expected to transform the quality of mathematical activity in the classroom, and at modelling the resulting process; in this perspective, the teacher’s acting in the classroom is strictly intertwined with analysing both motives and conditions for transformation.

3. THEORETICAL FRAMEWORK

3.1. The Function of Frameworks in Research for Innovation: Frameworks as Kit of Tools

We do not aim at giving in this paper an organic presentation of the framework of our research; even the word framework could suggest, against our
intention, a static structure that gives shape and support in advance to the classroom experiments. A better metaphor for research for innovation is a *kit of tools* (Wertsch, 1991), as we allow ourselves to borrow tools from heterogeneous and even competing theories, in order to cope in the most powerful and efficacious way with the design and the modelling of different school processes (see the discussion on *complementarity* in Steiner, 1985; Bartolini Bussi, 1994a).

In the following, for the sake of presentation, we shall adopt a top-down format, i.e. from the theoretical framework to design and analysis. However, we wish to emphasise that the *actual process was dialectical*; sometimes the framework was a guide for designing, but in most cases it was a way to make explicit the underlying assumptions of the classroom processes that had been designed and tried out on the basis of deep yet unconscious beliefs of the members of the research team.

3.2. *Inside the Kit of Tools: Learning and Teaching*

In the Vygotskian tradition learning is not considered a direct but a mediated relation between individuals and knowledge: the process of *semiotic mediation* is described by Vygotsky (1978) as the active introduction, from outside the subject, of an intermediate link (e.g. a sign) between the stimulus and the response which constitute the elementary forms of behaviour. Intermental functioning, that appears between people and may be used to determine the *zone of proximal development* for individuals, is a precursor of intramental functioning that appears within individuals. Hence learning cannot be separated from teaching. The development of *consciousness* has its origins in the cultural forms of social interaction (Vygotsky, 1982). The key to understand all the mental processes of individuals is *internalisation* (Vygotsky, 1978). Theoretical constructs such as *zone of proximal development*, *internalisation* and *semiotic mediation* are part of the kit of tools for our research team.

3.3. *Inside the Kit of Tools: the Study of the Long Term Teaching-Learning Process*

The process of internalisation can be studied in either laboratory or classroom setting: while the former is influenced by the research rules, the latter has the constraints as well as the freedom of the school institution. Typical of the school institution is the length of the observable processes. A three year length for a teaching experiment is consistent with the expectations of most Italian teachers, who teach the same class for the whole primary school (five years), as well as with the text of official school pro-
grams which underlines the importance of short term as well as long term processes.

The theory of activity, actions and operations developed by Leont'ev (1978) is supposed to offer a suitable tool to either differentiate or coordinate the analysis of long term and short term processes. We refer to the original formulation of the theory that has sometimes been applied to mathematics education (e.g. Christiansen and Walther, 1986; Mellin-Olsen, 1987) rather than to the documented implementation of it in the Soviet Union, that has often been criticised (see Streefland, 1991).

The basic distinction is between activity, actions and operations. The first level is the global level: the collective activity corresponds to a motive (or to several motives in the most common case of polymotivated activity). The term motive has a specific meaning for Leont'ev; it is the object (either a concrete or a mental one) that can meet a need of the acting subjects. The famous example that is provided by Leont'ev (1981) is the primeval collective hunt, whose motive is given by food or clothing (or both). However the motive cannot be directly recognised in the way of acting of any participant; a beater, for instance, aims at frightening a herd of animals and sending them towards other hunters. His action, without consideration of the overall collective activity, seems senseless and unjustified. It could even seem senseless to the beater himself, when he starts to imitate the old members of the group. This example shows that the motive of the collective activity could be hidden for some of the participants.

The activity we consider is the teaching-learning activity: it is carried on collectively by all the participants in the life of the classroom (the teacher, the pupils and others who enter the classroom).

Human activity exists only in the form of action or a chain of actions (Leont'ev, 1978). Actions are related to conscious goals. Frightening animals is one of the actions, by means of which the activity of hunting exists (Leont'ev, 1981).

Actions are carried out in variable concrete circumstances, by means of operations, which directly depend on the conditions of attaining goals in specific situations. For instance a beater can go around an obstacle if it occurs.

All the participants in the teaching-learning activity act and operate in some way. We are especially interested in the teacher's actions and operations; among the former there are the tasks, among the latter there are the communicative strategies. The study of short term processes concerns the relationships between actions and operations, while the study of long term processes concerns the overall activity.
3.4. Inside the Kit of Tools: Meaning and Personal Sense

The motives of this teaching experiment are manifold and concern both the mathematical concepts to be built and the quality of classroom interaction. Some of the motives of the teaching experiment can be related to meanings, using this word after Leont’ev: “Meaning is the generalisation of reality that is crystallised and fixed in its sensuous vehicle, i.e. normally in a word or word combination”. The genesis of meaning is social and historical: “[Man] assimilates the experience of preceding generations of people in the course of his life [...]. Meaning is thus the form in which the individual man assimilates generalised and reflected human experience” (Leont’ev, 1981, p. 226).

How meaning relates to individual consciousness is the basis of the definition of personal (or individual or conscious) sense. The sense (of something) is a relation between the subject and the world that is created by the subject’s activity on something. It is determined by the relation between what stimulates the subject to act and what his action is directed to: “The conscious sense expresses the relation of motive to goal [...]. The question of personal sense can thus be answered by bringing out the corresponding motive” (Leont’ev, 1981, p. 229).

At the very beginning of every teaching experiment, some motives are supposed to be known by the teacher only. One of the aims of schooling is fostering the expression of pupils’ personal senses in meanings, that is establishing an explicit relationship between pupils’ personal consciousness and existing objective knowledge.

3.5. Inside the Kit of Tools: Mathematical Discussion

A relevant feature of the Vygotskian approach is the emphasis on interpersonal processes as a basis for intrapersonal processes. A promising context (even if not the unique one) for the development of interpersonal processes seems to be classroom discussion (a review of the related literature is in Bartolini Bussi, 1991: however our approach seems to be different from others because of the focus on the long term processes and of the emphasis on the teacher’s role).

Mathematical Discussion is a polyphony of articulated voices on a mathematical object (e.g. a concept, a problem, a procedure, a structure, an idea or a belief about mathematics), that is one of the motives of the teaching-learning activity. The term voice is used after Wertsch (1991), following Bakhtin, to mean a form of speaking and thinking, which represents the perspective of an individual, i.e. his/her conceptual horizon, his/her intention and his/her view of the world. As such it is related to the membership of a particular and social category.
A form of mathematical discussion is the scientific debate that is introduced and orchestrated by the teacher on a common mathematical object in order to achieve a shared conclusion about the object that is debated upon (e.g. a solution of a problem). In this case the teacher utters a voice that represents the mathematical culture: the perspective on the object that is introduced by the teacher is usually different from the ones that are introduced by the pupils. In the classrooms that have already had a long term experience of such a debate, different voices are sometimes uttered by the pupils themselves; in the extreme case of individual tasks which incorporate previous collective activity, different voices are internalised (Vygotsky, 1978) and are uttered by an individual actor, who is overtly speaking to oneself. Elisa's protocol (§ 1) offers a number of instances of internalisation: the alternation of voices is sometimes explicit (see the quotation from Costanza and Piero della Francesca) and sometimes implicit (see the continuous shift from personal to impersonal – and general – speech; or the use of inverted commas to recall the utterances from either classroom discussions – [4, 5, 6] – or Piero’s paper as in the case of the word “degrade” [2]).

Mathematical discussions, as ways of realising the motives of the teaching-learning activity, are conceived as actions (Leont’ev, 1978). In this frame, teachers’ operations are the communicative strategies that are used in the debate (not only utterances, but also gestures, drawings and so on). The research on classroom communication (e.g. Edwards and Mercer, 1987) has shown that the management of classroom interaction is not trivial, because of the emergence of unsuspected patterns and routines. To face this problem, some results of the research project on Mathematical Discussion in Primary School are assumed in the kit of tools of the teaching experiment. They concern the concept of discussion itself, the main models of classroom discussions (e.g. negotiating or “balance” discussions and conceptualisation discussions, see Bartolini Bussi, 1991) and the detailed study of the communicative strategies that can realise teachers’ intentions in a given context. They constitute the embryo of a didactical theory of discussion that is under construction by means of the teaching experiments. However in this paper we shall address only the level of activity-actions (the complementary analysis that considers also operations is exemplified in Bartolini Bussi and Boni, 1995).

3.6. Inside the Kit of Tools: the Field of Experience

The length of the experiment was expected to shape very deeply pupils’ image of mathematics, and mainly of geometry. We decided to carry out experiments within a field of experience (Boero, 1992), where problem
generation is determined not so much by the present systematisation of mathematics (e.g. solving isolated problems about geometric transformations) but rather by the activity of exploration itself (e.g. studying the problem of plane representation of the three-dimensional world by means of mathematical tools): the emphasis is on mathematics as a cultural and historical process, where different strands were intertwined to solve problems from the physical, social and mental world.

This choice is consistent with Freudenthal’s (1983) arguments in support of didactical phenomenology of mathematical objects: the didactical approach to a part of geometry is determined by the experience of the phenomena that have been thus organised by mankind rather than by the experience of the results of such an organisation.

4. FIRST ANALYSIS: THE MOTIVES OF THE TEACHING-LEARNING ACTIVITY

The elicitation of motives (Leont’ev, 1978) is a fundamental step in the design and the analysis of the teaching-learning activity. In the following, we shall outline the main motives of the teaching experiment. In the last three columns of Table 1 some of the motives are recalled and related to the specific actions.

4.1. Drawing: Pupils’ Needs

There is a wide literature on child drawing, considered as a process-product of external representation of space in cognitive tasks (e.g. Freeman, 1980). In most of the literature on spatial abilities, drawing and image reading are considered a diagnostic tool (see Hershkowitz et al., 1990). The Vygotskian approach emphasises also the functional role of drawing and image reading in the overall development of the child (cognitive, emotional and so on); as such, drawing is not treated as isolated from other mental abilities of the child (e.g. oral and written forms of language, Stetsenko, 1995).

The loss of engagement in spontaneous drawing as soon as children master a new tool of external representation, as durable as drawing (i.e. written language) can be related to the critical analysis of their own products (Vygotsky, 1972), that are judged as no longer suitable to represent what they see.

A motive of the teaching learning activity is a set of tools for drawing that can meet the increasing need of likely representation of the visible world.

At the very beginning, this motive was actually the most (maybe the only) conscious one for the pupils, because they were driven by the need of
### TABLE I

<table>
<thead>
<tr>
<th>Classroom</th>
<th>Sem. med.</th>
<th>Social construction of knowledge</th>
<th>Epistemolog. complexity</th>
<th>Math. reasoning as a chain of statements</th>
</tr>
</thead>
<tbody>
<tr>
<td>A/B/C</td>
<td></td>
<td>The table and the small ball</td>
<td></td>
<td></td>
</tr>
<tr>
<td>/B/C</td>
<td></td>
<td>*</td>
<td>*</td>
<td>(●)</td>
</tr>
<tr>
<td>A/B/C</td>
<td>M1</td>
<td>Geometrical representation of experiential spaces</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A/B/C</td>
<td>M2</td>
<td>*</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>A/B/C</td>
<td></td>
<td>From micro-space to meso-space</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A/B/C</td>
<td></td>
<td>*</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A/B/C</td>
<td></td>
<td>*</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>A/B/C</td>
<td>M3</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>A/B/C</td>
<td></td>
<td>Individual comment</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A/B/C</td>
<td></td>
<td>*</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>A/B/C</td>
<td></td>
<td>*</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A/B/C</td>
<td>M4</td>
<td>*</td>
<td>*</td>
<td>(●)</td>
</tr>
<tr>
<td>A/1</td>
<td></td>
<td>The rectangular floor</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A/1</td>
<td>M5</td>
<td>*</td>
<td>*</td>
<td>(●)</td>
</tr>
<tr>
<td>A/1</td>
<td></td>
<td>*</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>A/1</td>
<td>M6</td>
<td>*</td>
<td>*</td>
<td></td>
</tr>
</tbody>
</table>

* Main motives of activity.

↓↓ Actions with conscious goals.
drawing better. In the following we will focus on the motives which determined the particular teacher’s actions and operations in the classroom.

4.2. Joint Activity: the Social Construction of Knowledge

The Vygotskian tradition emphasises the social construction of human knowledge. This conception is related to a whole philosophical trend of European culture which was explicated as from the 16th century (e.g. by Bacon, Pascal, Galileo) when the concept of scientific progress was explicitly linked to the kind of social construction that no longer considered the ancient texts repositories of truth but rather partial steps in the building of knowledge (Rossi, 1971).

A motive of the teaching-learning activity is the culture of scientific progress by means of the social construction of knowledge.

4.3. Perspective: Its Function in the Historical Development of Geometry

The function of the theory of linear perspective in the development of modern mathematics is well known: it was the root of projective geometry and of the study of geometric transformations (see for instance the historical reconstruction of modern geometry in Lehmann and Bkouche, 1988 and the phenomenology of descriptive geometry in Freudenthal, 1983).

Perspective forces us to consider the dialectical relationships between mathematics and other fields of human activity such as painting, architecture and technology. The development of perspective, even before its complete mathematisation, is strictly intertwined with the development of a mathematical model of vision and with a theory of infinite and homogeneous space, which are different from the ancient ones and which are the basis for the modern attitude towards space. In this sense, perspective can be assumed, after Cassirer, as a “symbolic form, i.e. a concrete perceptible sign connected to a peculiar spiritual content and intimately identified with it” (as quoted in Panofsky, 1961: p. 50). Hence the epistemological complexity that has to be considered in school mathematics (Hanna and Jahnke, 1993) is well embodied by such a field of experience where the relationships between reality and its mathematical models are continuously brought up for discussion.

A motive of the teaching learning activity is the epistemological complexity of geometrical experience.
4.4. Perspective: Its Function in Teaching Geometry in Primary School

There are good reasons to introduce perspective in primary schools, even if it is not usually considered in the core of the geometry curriculum for young learners (see Freudenthal, 1983). We shall consider the primary school programs in force in Italy since 1985. They read: The curriculum of elementary geometry, tending towards the organisation of the spatial experiences of the child, will be developed through the progressive introduction of physical reality; by the study and construction of models and drawings, a knowledge of the principal plane and solid geometric figures and their elementary transformation will be attained (Barra et al., 1992, p. 57). By means of perspective drawing, pupils cope with a whole network of concepts such as length; angle; horizontal and vertical lines; parallel, intersecting and perpendicular lines; elementary transformations (which are all quoted in the programs) that refer to both three-dimensional and two-dimensional geometry: they are not alternative words to describe figures and configuration but necessary tools to solve complex representation problems.

A motive of the teaching learning activity is a network of geometrical concepts for three-dimensional and two-dimensional geometry as well.

The recourse to the field of experience of perspective drawing meets also the need of balancing the emphasis of measuring experience in primary school. The Italian programs underline the function of measuring as a cognitive tool that increases the possibility of understanding facts and phenomena not limited to the fields of length, weight and areas. Yet the contribution of geometry to knowledge cannot be reduced to measuring. Rather, when measuring is developed in connection with natural science as a means to transform rough estimation into a “scientific” procedure, it tends to give the greatest value to the empirical comparison between the text (e.g. hypothesis, estimate and so on) that has been produced and the facts that are perceived or observed. The danger is in conveying the implicit message that validation in mathematics stems from matching texts and facts. This is obviously not true, but a different belief, that gives value to deductive reasoning too, cannot be conveyed without specific activities.

A motive of the teaching learning activity is geometrical proof in the form of a chain of statements.

4.5. A Provisional Conclusion

On the basis of the above arguments, we claim that the activity in the field of perspective drawing must not be conceived as a kind of enrichment activity to be reserved to high level classrooms, because the motives
of the teaching learning activity in the field of perspective drawing are fundamental elements of mathematical experience.

5. SECOND ANALYSIS: FROM ACTIVITY TO ACTIONS

In the previous § 4 we have elicited the motives of the activity. Yet no motive can be taught directly, as activity exists only by means of actions. By designing and analysing activity through actions, the cyclic time of activity is transformed into the linear time of the chronicle of actions (Engeström, 1991).

The link between activity and actions is very strong. For instance, the motive of social construction of knowledge (§ 4.2) contains either the critical recourse to the existing culture or the practice of collective activity in the classroom.

Hence the design of our teaching experiment contains not only individual tasks, but also mathematical discussions and appropriation of existing cultural artefacts (e.g. devices, texts and so forth).

In the following we shall provide a sketchy chronicle of actions, before discussing in more details some crucial issues. The chronicle of actions defines the sequence of the rows of Table 1.

The main reference will be to the three classrooms which developed the majority of the experiment: the first classroom (classroom A; teacher: Franca Ferri) carried out the experiment from the end of the 2nd grade to the end of the 5th grade (i.e. 1990–94), while the second and third classrooms (classroom B; teacher: Mara Boni; classroom C, teacher: Silvia Salvini) carried out the experiment from the beginning of the 3rd grade to the end of the 5th grade (i.e. 1990–93). We shall see in the following that the development of the experiments was not exactly the same, as the standard scheme was adjusted to the needs of each classroom (see also the first column of Table 1).

At the end of the 2nd grade the pupils were able to draw simple three dimensional objects (on the base of the activity of coordination of spatial perspectives, Bartolini Bussi, in press), such as set of boxes, tables and chairs with a mastery that was surprising for the age level. By means of photo analysis they had discovered also the existence of some vanishing points on the horizon, but the achievement was still fragile, mainly based on perception. Verbalisation was ambiguous and difficult because they lacked different words to refer to objects and to images (drawings). For instance they discussed the legs of the table referring both to the concrete table and to an image (either drawing or photo) of the table. Very often they had to help themselves with gestures to produce less ambiguous utterances.
5.1. *The Problem of the Table and the Small Ball*

A problem (Figure 2) was given in the school year 1990–91 in the 2nd (classroom A) and the 3rd grade (classrooms B and C): a perspective drawing, that represented a table with a small ball beside on the floor, was given together with the task: *Draw the small ball in the centre of the table. You can use instruments. Explain your reasoning.* In the problem there was something old (the tradition of drawing tables was well established) and something new (the explicit task).

The strategies from the classrooms can be roughly separated into three classes:

(a) rough estimate by sight, i.e. the ball is drawn directly on the table, without comment;

(b) measure-dependent: the ball is drawn in a point, that is chosen after measuring (or finding medium points, by means of symmetry, paper folding and so on) and tracing lines (the majority of pupils);

(c) alignment-dependent: the point is chosen as the intersection of diagonals.
5.2. *Discussion: Empirical Solution*

In classroom A the teacher decided to avoid a negotiating discussion (see balance discussion in Bartolini Bussi, 1991), because of the young age of pupils and of the classroom tradition which permitted waiting a long time before having the problem corrected.

In the classrooms B and C a negotiating discussion was initiated by the teacher. The difficulty of coming to a shared correct solution was evident: in no case did even those with correct strategies succeed in arguing for their solution to the extent of convincing their schoolfellows. Rather some of them started to doubt their own solution.

The problem was solved by the teacher empirically. Four tables in the classroom were prepared: (a) in the first the top was clear; (b) in the second, the diagonals were drawn on the top (Figure 3); in the third the median-lines were drawn (Figure 4); in the fourth both diagonals and median lines were drawn. Pictures of the tables were taken with a camera so to have images similar to the one of the previous problem. Then each pupil traced the pictures with transparency film, to repeat on a model the process of perspective drawing, discussed the images (both photos and drawings), observed that the middle points of the sides of the represented table did not represent the middle points of the sides of the real table and agreed that the right solution was the diagonal one. This empirical solution was institutionalised by the teacher. Pupils gave a lot of justifications: *The far pieces become smaller. Perspective changes the lines. From reality to image things change.* The pupils focused on the changes that had happened in the drawing.

However, tracing pictures offered an empirical verification, but not a theoretical validation of the solution, which could have linked the strategy to other pieces of established knowledge. Without such a link every new problem would have been solved empirically against the tradition of the classrooms of gaining general methods to be applied to entire classes of problems. The following steps were designed to address this issue.

5.3. *The Geometrical Models of Experience Spaces*

Some problems were designed concerning: (1) the geometrical description of reality in meaningful situations (e.g. the description of serving phases in volleyball; the description of body positions in shadow theatre); (2) the geometrical description of images of reality, distinguishing between the conventions that are used in drawing, in writing, in map-making and so on. The geometrical knowledge which had been assumed as a basis was the one developed outside the teaching experiment by means of a standard paper.
and pencil activity, where geometrical terms had been used to describe figures and configurations.

The first set of problems aimed at suggesting a geometrical model of the space of everyday experience. The second set of problems aimed at suggesting a geometrical model of the sheet: paper and pencil is a more
traditional setting for standard geometrical work, but it was necessary to deautomatise a lot of implicit conventions that had been used by the pupils, such as the different frames of reference that were used in either writing or drawing or making a map on a sheet of paper.

Both sets of problems were carried out in the same weeks, alternating individual tasks and collective discussions (Ferri, 1993a): from then on the pupils succeeded in gradually detaching themselves from objects and images and in focusing on the geometrical properties of their shapes, avoiding the ambiguity of using only the same concrete terms (e.g. the legs of the table) for both. The pupils allowed themselves to use the same words for the sake of brevity, but they were conscious of the convention (see the use of inverted commas in the sentence [3] of § 1).

5.4. Invariants in Perspective Drawing

As soon as the pupils had partially appropriated suitable geometrical terms to describe geometrically either three-dimensional objects or two-dimensional images, the time came to return to the relationship between reality and representation between objects and images or between three-dimensional and two-dimensional figures. A two-column scheme was built in a discussion to point out the geometrical properties of three-dimensional figures that were changed or unchanged (i.e. the invariants) in their two-dimensional perspective representations (Ferri, 1993a; Bartolini Bussi, 1994b). In the left column there was reality, in the right column representation (or sheet). For each geometrical property that was known by pupils (e.g. parallelism, alignment, number of sides of a polygon, and so on) they made some examples and wrote changed or unchanged.

5.5. A Conflict: from Microspace to Mesospace

The previous scheme proved to be very useful with problems of microspace either to direct or to check the process of drawing. Yet when a problem concerning mesospace (Berthelot and Salin, 1992) was posed (real life drawing of either the corridor or the classroom) the pupils were overwhelmed by the complexity of strategic decisions about the representation of the whole space that contains objects as well as the pupils themselves. All the pupils had to write also an individual comment on their own drawing. Some embryos of geometrical solutions by means of special lines and points were offered by some pupils (either implicitly in the process of drawing or explicitly in the individual written comments) with no support but drawing likeness: they were attempts to represent parallel lines as lines that converge in the points of a zone that is close to the centre of the picture
(this technique is well documented in painting up to the middle of the 15th century, Panofsky, 1961).

5.6. Discussion: the Need of a General Method

In the discussion different drawings were compared in order to focus on the ways of representing the critical parts: vertical lines on the wall; the ceiling; the perceived length of far objects. The embryonic geometrical solutions were shared among the pupils, who tried to analyse their own products by means of them. However, even if some drawings looked more likely than others, the need for general solutions of the problems of mesospace representation was explicitly stated in all classroom discussions.

5.7. A Reading from Piero della Francesca

Two excerpts from De Prospectiva Pingendi (Piero della Francesca, 1460) were introduced into the classrooms for collective reading and interpretation. They concern:

(1) the theory of vision and painting, where the main elements for a systematic approach to painting are introduced (i.e. the eye; the shape of what is seen; the distance between the eye and what is seen; the lines between the ends of what is seen and the eye; and the place, that is between the eye and what is seen, where things are to be put);

(2) the first step in the drawing of a square grid on a “degraded” square of the ground plane (e.g. the image of a floor of a square room), where the lines that are perpendicular to the picture plane are represented as lines that converge into the principal or vanishing point (Figure 5): the second step, i.e. drawing the “degraded” lines that are parallel to the picture plane, was not introduced.

In the texts there was something old (the comments on painting and vision, that were similar to the ones of the pupils) and something new (the method to solve a difficult problem).

5.8. Individual Comment

Because of the old-fashioned language (Italian of the 15th century), the teacher had to act as an expert guide in interpreting the text. The reading was carried out successfully in all the classrooms. As usual the pupils were requested to comment individually in writing this task, answering the question what did you understand? The initial protocol (§ 1) is taken from this experience in the classroom A.
5.9. *The Problem of the Table and the Tablecloth*

A new individual problem was given (Figure 6). A table was drawn in perspective together with a squared tablecloth that was drawn as a rectangle. The task was *draw the tablecloth on the table*. In the problem there was something old (the tradition of drawing tables, the implicit reference to the method of Piero for lines perpendicular to the picture plane) and something new (the point of view of the drawing that is not in front of one of the sides of the table, the need of a new method to represent the lines that are parallel to the picture plane).

5.10. *Discussion: What is the Shape of the Table?*

Most of the pupils interpreted the problem as if the tablecloth were adhesive: hence they did not consider the problem of the leaning parts. Most of the pupils gave a partial correct solution and succeeded in applying the method of Piero to the lines that are perpendicular to the picture plane (the 'vertical' lines of the tablecloth in the drawing). In the discussion all the pupils agreed on the use of the method of Piero; on the contrary, the way of representing lines parallel to the picture plane was debated very carefully, as most of pupils had proposed empirical solutions (similar to the ones that were used before the complete mathematisation of the perspective method, Panofsky, 1961). The solution of Figure 7 (where a diagonal is used as an intermediate step) was accepted for a while as the right one. Yet when the teachers (as designed in advance) posed the question: *what is the shape of the table?* The pupils became immediately aware that posing a 4 times 4 tablecloth on the top of the table had stated implicitly that it was a square table, in spite of the appearance and of the most common experience. In this case, as $8 \times 4 = (4 + 4) \times 4$, a shared final solution was obtained.
by dividing the table into two squares (Figure 8) and by using the whole tablecloth to cover the top of the table.

5.11. *The Rectangular Floor*

The last part of the teaching experiment was actually tried out in the classroom A only, in the school year 1993–94 (5th grade). We shall give only a little information, to show the complete design of the sequence of actions. Two problems were posed. In both the contour of a rectangular floor was already drawn as a trapezium in perspective; the map of the floor paved with square big tiles was also given (5 times 7 in the first
case; 7 times 5 in the second case, with the first number relating to the number of tiles in front of the observer). The task was: *Draw the tiles on the floor. Explain your reasoning.* The problems contained something old (the method of vanishing points) and something new (the measures of the sides of the rectangle, that are prime to each other).
5.12. *Discussion: the Method of Jean Pélerin*

In the discussion the strategies and the solutions were compared, in order to build the method of distance points by Jean Pélerin (Lehmann and Bkhouche, 1988; Kemp, 1990), that is the general solution of paving with square tiles a rectangular floor with two sides parallel to the plane of the picture.

5.13. *A New Perspective on the Table and the Small Ball*

The same problem (see § 5.1) was posed individually with a different formulation of the task. Instead of *explain your reasoning*, pupils were asked to *prove your method to find the centre*. The words *prove* and *proof* (the Italian *dimostra* and *dimostrazione*, that are used for mathematical proofs) had been used previously by the pupils in the discussions in a naive way, to mean a very convincing argument. The aim of the problem was to study to what extent the familiarity with verbal arguments during the whole experiment (e.g. discussions) and with early geometrical arguments in some special steps (e.g. § 5.4 and § 5.7) could have induced pupils to produce a mathematical proof, in spite of the difference between argumentation and proof (Balacheff, 1991; Duval, 1991). All the pupils solved the problem correctly and eleven pupils out of nineteen succeeded in producing a chain of statements that explicitly related the result to the invariance of alignment in the shift from three-dimensional space to two-dimensional plane, by means of perspective (five more pupils gave an incomplete justification by means of rejecting some wrong strategies, see Costa *et al.*, in press).

5.14. *Building Space Before Objects*

A worksheet (taken from a high school textbook on geometrical drawing) was given to the pupils for individual reading. In the worksheet the method for drawing a room (not only the floor, but also three walls and the ceiling) was described without any justification. The excerpt contained something old (the square floor of the room, the problem of representing mesospace) and something new (the method to coordinate the drawing of the floor, the walls and the ceiling, the suggestion to locate objects). The pupils were asked to follow the instructions and to make an individual drawing, putting also some furniture inside the room, as they liked.
6. INSIDE THE CHRONICLE: TOOLS FOR SEMIOTIC MEDIATION

In this section we shall go through the whole activity once more. The guiding thread will be offered by the most relevant examples of semiotic mediation (§ 3.2).

In the second column of Table 1, emphasis is put on those steps where the class is explicitly given a tool for semiotic mediation.

6.1. Geometrical Representation of Experiential Spaces

Pupils' needs can be reconstructed going through the previous steps of the teaching experiment. Pupils were acquainted with photographs, which showed clearly some perplexing features of perspective representations; nevertheless this was not enough to accept the same features in drawing. The first problem of the table and the small ball (§ 5.1) was designed to be very troubling. It contained a number of potential conflicts (e.g. everyday vs. school setting; perception vs. knowledge, reality vs. representation, precision vs. truth). In the discussion the most intriguing fact was the total impossibility of keeping the interaction on only one plane: hints from competing positions were continuously mixed.

As we have seen in the chronicle a shared solution (in the classrooms B and C only) was found by means of pictures (§ 5.2). However they offered an empirical verification of the diagonal solution, but not a theoretical validation of it, which could have linked this result to other pieces of theory. In other words, pupils were able to build texts describing the experiment itself but not to build any local theory of perspective which could have given to the diagonal solution the status of a theorem derived from other more basic and general statements (e.g. in perspective representation, the intersection of the diagonals of the quadrilateral that represents the rectangular top of the table is the image of the centre of the top of the table; justified by means of the explicit statement: every straight line connecting two point is represented by the straight line connecting the two corresponding points or something similar). Without such a theory every new problem would have been solved again empirically against the tradition of the classrooms of gaining general methods to be applied to entire classes of problems.

Pupils were not expected to be able to generate by themselves such general methods as research on real life drawing clearly shows (Freeman, 1980). Moreover the history of perspective shows that early handbooks of perspective drawing were rather series of recipes for either constructing a convincing architectural setting or foreshortening such stock elements as cubes, polygonal well-heads, column capitals and human figures (Kemp, 1990). Only when the geometrical theory of perspective was elaborated,
were the problems of real life drawing completely solved. Yet the theorisation of perspective was possible only when a theory of vision and painting and a theory of space, different from the ancient ones, were built in the 15th century (Panofksy, 1961). Identifying this epistemological obstacle gave the guidelines for further classroom activity.

So the individual task of describing some real life situations or some images by means of geometrical terms was a first example of semiotic mediation, where the spontaneous pupils' behaviour (i.e. using everyday language or maybe expressions referring to emotions) was inhibited by the teacher's request. From then on the pupils were aware that the verbalisation about the strategies of real life drawing should have shifted to a different level and should have been expressed by means of the geometrical properties of the figures that represented the objects as well as the images.

6.2. The Scheme of Invariants in Perspective Drawing

The two-column scheme (what is changed, what is unchanged from reality to sheet, § 5.4) was built collectively in a discussion orchestrated by the teacher. It was understood as a way to shift from the naive attitude (§ 5.2) of observing only what is changed (e.g. lengths, parallelism, angles and so on) to observing what is unchanged (e.g. alignment, the number of sides of a polygon and so on) in perspective drawing. This shift is documented in the history of perspective too, by means of the work of Desargues, who first stated a symmetrical relationship between the object-figure and the image-figure and studied their invariants instead of the "degradation" of objects into images (Field and Gray, 1987).

From then on the scheme acted as a semiotic tool in perspective drawing for either producing or reading an image: some pupils were able immediately to recur to the scheme themselves; some pupils had to be solicited by the teacher. In any case, the presence (or the evocation by memory) of the scheme created an intermediate link between the stimulus (the observation of either an object or a given image) and the response (either the production of a drawing or the reading of the image); when the scheme was drawn into the operation, the simple stimulus-response process was replaced by a more complex process where the direct impulse to react was inhibited from the outside.

6.3. The Method of Piero della Francesca

In this case an historical source was directly introduced into the classroom by the teacher. Actually the status of the two excerpts was different. The first one, concerning the elements of the theory of vision, was a way of
expressing pupils’ personal senses into (historically documented) meanings (§ 3.4). Hypotheses about the mechanism of vision had appeared from time to time in the discussions to argue in favour of or against some way of representation. Piero della Francesca had given a brief account of it, lending pupils common words to express thoughts similar to their own. Hence, the first excerpt is a semiotic tool for the cultural formation of consciousness. The second one, that concerns the drawing of a square grid, is a semiotic tool that inhibits the empirical search for solutions. Moreover their joint reading offers a model of the genesis of a method of solution; hence history is not a store of ready tools but rather a set of mechanisms to develop new tools, according to the European philosophical tradition inherited by Vygotsky.

6.4. The Other Semiotic Tool for Perspective Drawing

In the last part of the experiment, more refined semiotic tools for perspective drawing were created or appropriated by the pupils:

(1) the complete method for the square grid on a square (§ 5.10) and the method for a rectangle (§ 5.12) were built collectively in the discussion, by sharing some individual solutions;

(2) the method for building the interior of a room (§ 5.14) was appropriated from a textbook, where it had been presented in a totally decontextualised way.

6.5. Semiotic Mediation or Semiotic Mediations?

The above examples clearly show that the ways of realising semiotic mediation in the classroom are manifold:

(1) a tool that has been developed elsewhere (paper and pencil setting) is drawn into new problems with the effect that pupils’ personal senses of geometrical concepts are also enriched (§ 6.1);

(2) a new tool (the table of invariants) is created collectively through a discussion orchestrated by the teacher (§ 6.2);

(3) two new tools are introduced simultaneously by the voice of Piero della Francesca: the first to develop personal senses into meanings, as concerns the theory of vision and painting; the second to inhibit approximate search of converging lines in perspective (§ 6.3);

(4) two tools that have been created by individual pupils are shared in the discussion (§ 6.4);
(5) a tool from the existing knowledge (an excerpt from a school textbook) is read individually (§ 6.4).

A further model of classroom functioning of semiotic mediation is given by the following example. An important device from the history of perspective is the perspectograph. A simple perspectograph is an instrument that consists of a glass sheet and of an eye-piece. The sheet is between the eye-piece and what has to be drawn. The painter looks through the eyepiece and copies by transparency what is beyond the sheet. Perspectographs were very common during the 15th–18th centuries in painters’ workshops (Kemp, 1990). They have a twofold function. On the one side, they are practical instruments, that were used by painters to draw by transparency. On the other side they are theoretical instruments that embody the theory of vision and painting: the shape on the picture plane is obtained by intersecting the visual cone (or pyramid, as the ancients theorists preferred to say), that is constituted by the visual rays that go from the eye of the painter (the vertex) to a whichever point of the object is to be drawn.

In classrooms B and C some cardboard and plexiglas perspectographs had been built by the pupils themselves and were used in the course of the empirical verification in alternation with pictures. In classroom A this step was not carried out. However a pupil of the classroom, Costanza, had seen a wood and glass model of a perspectograph and had been taught how to use it by her parents. She explained in detail the artefact, making also a small card model of it during her comments on the individual drawing of the classroom. Many of her schoolfellows were struck by her description and twelve (out of nineteen) used the perspectograph to interpret and illustrate the reading from Piero della Francesca and the further problems as well. The example of the perspectograph was immediately picked up by the teacher who contextualised the perspectograph in the 16th and 17th centuries by means of some Dürer’s xylographies and by means of excerpts from the picture *The Draughtsman’s Contract* (Greenaway, 1982), where a painter systematically uses this device to draw the building in the gardens of Compton House. In this case the role of guide in the classroom was initially played by the girl, who had received assistance, in turn, from her parents, and later by the teacher who transformed an anecdote into a true piece of classroom activity.

7. CONCLUSIONS

The complete analysis of the classroom process (individual protocols and discussion transcripts) is far beyond the scope of this paper (exemplary
analyses are in Ferri, 1993b; Costa et al., in press). We can only draw some conclusions about the results of the experiment and their reproducibility.

A limited amount of school hours (between 15 and 20 distributed over three years) were used with the following main results:

(1) in all the classrooms, the mastery in real life drawing and in image reading was exceptionally improved;

(2) in all the classrooms, epistemological complexity was in the foreground; a network of geometrical concepts was explored by the pupils in order to model the problem of plane representation of three-dimensional space;

(3) in classroom A, that completed the experiment, examples of mathematical reasoning in the form of a chain of statements, without needing to refer to concrete referents, were produced by most of the pupils.

The three above points suggests it is appropriate to relate this study to at least three different research issues on the teaching and learning of geometry:

(1) visualisation;

(2) the development of geometrical units in the curriculum;

(3) the early approach to geometrical proofs.

As for the point (1), the set of the tasks of the teaching experiment is an example of training of the visualisation ability, i.e. the ability to represent, transform, generate, communicate, document and reflect on visual information (Hershkowitz et al., 1990). Such examples are not very frequent in the literature (relevant exceptions are discussed below): in the Piagetian tradition, drawing is often used for diagnosing spatial abilities rather than for training spatial abilities. On the contrary, in the Vygotskian tradition, drawing is given a functional role in the overall development of the child, to master the ability to represent the world symbolically in decontextualised ways (Stetsenko, 1995). This study belongs to this tradition; besides oral (e.g. discussions) and written (e.g. individual comments) forms of language, other cultural signs are introduced, namely the early methods of perspective drawing.

As for point (2), the few teaching experiments that introduce drawing of objects in the geometrical curriculum usually refer to technical drawing in the form of parallel perspective (see Yakimanskaya, 1991; Bessot and Verillon, 1993) and concern older students. The teaching experiments of
the so-called realistic mathematics (Freudenthal, 1983; Treffers, 1978) introduce the phenomenology of linear perspective as a basis for geometry in primary school too. The teaching experiment we have presented here has some similarity with them but, in addition, it introduces an explicit historico-cultural dimension in the classroom, according to the Vygotskian hypothesis of the cultural development of consciousness. Actually the pupils become conscious of their own intellectual processes by locating themselves in the history of the self, of their classroom and of their social group.

Point (3) above confirms the feasibility of the objective to get young pupils constructively involved in approaching the problem of geometrical proof in suitable educational contexts, that contains particular and concrete culturally based geometrical situations (Boero and Garuti, 1994), in opposition to the extended literature on the poor proof-making ability (Hershkowitz et al., 1990; Clements and Battista, 1992). Further teaching experiments are needed to confirm the results from this study. A replica of the whole experiment is now in progress in other classrooms ranging from grade 3 to 7. The data from the early two steps confirm

(1) the distribution of the solutions of the problem of the table and the small ball (§ 5.1) (see the first lines of Table 2), that seems to be independent from pupils' (or students') age (see the last line of Table 2);

(2) the emergence of the same constellation of problems (§ 5.2) in the discussion (up to even University level).

The general motive of the teaching learning activity that characterises the classroom life, in either the already complete or the still incomplete experiments, is the social construction of knowledge (§ 4.2). This motive is realised, on the one side, by the format of discussion, that gives value to joint activity in mathematical experience and, on the other side, by the systematic recourse to tools for semiotic mediations, that are explicitly drawn from the existing culture. Theoretically, in a Vygotskian approach the two things cannot be conceived without each other. Experimentally, the two things cannot be easily separated. The example of Costanza and the perspectograph (§ 6.5) is clear: Costanza introduced a new element in the collective activity; the element was accepted by her schoolfellows and further elaborated by the teacher. Even if Costanza was physically present in the discussion, she was uttering a voice from outside the classroom, i.e. the voice of the culture that has directed the joint activity with her parents. Sometimes the voice from outside can be uttered by an ancient document or by a school textbook. However the long acquaintance with the active discussion of schoolfellows or teacher's utterances put the pupils
TABLE II
The problem of the table and the small ball. Data on the ongoing experiments.

<table>
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<th>SUBJECTS</th>
<th>SOLUTIONS BASED ON</th>
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</table>

(1) the pupils can give more than one solution.
(2) established tradition of real life drawing.
(3) established tradition of discussing geometrical problems.
(4) students of the 3rd and 4th year of the course of Mathematics, who had passed at least two university examinations of geometry: they have studied projective geometry too by means of linear algebra.

in an active attitude: the methods of Piero and the methods of Costanza can be discussed and criticised in the same way (§ 1) and not passively accepted like repositories of truth. Hence mathematical discussion and semiotic mediation seem to be the two sides of a same coin both theoretically and experimentally. A further development of a didactical theory of mathematical discussion (see Bartolini Bussi and Boni, 1995) and a careful analysis of tools of semiotic mediation for activity in other fields of experience will hopefully allow the design and implementation of new teaching experiments in order to explore better the feasibility of this kind of social construction for other pieces of mathematical knowledge.

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