MATHEMATICAL DISCUSSION AND MATHEMATICAL UNDERSTANDING

ABSTRACT. A longitudinal study is being undertaken into the question of whether or not discussion in the mathematical classroom is an aid to understanding. Preliminary analysis shows that, while the hypothesis that discussion aids understanding is very attractive, evidence to support the hypothesis has been limited. Classroom observations during the first year of the study have led to a method of classification of mathematical talk in the classroom which we expect will prove useful in investigating links between mathematical discussion and mathematical understanding.

The ability to 'say what you mean and mean what you say' should be one of the outcomes of good mathematics teaching . . . . Pupils need the explicit help, which can only be given by extended discussion, to establish these relationships (between different mathematical topics); even pupils whose mathematics attainment is high do not easily do this themselves. (Cockcroft, 1982, England)

In all other topics, we saw teachers there (Kitamaeno School in Tokyo) using peer group dialogues . . . (they) justified their use of group dialogue because of their large classes (35-43) and the unusually independent spirit of many of the 600 pupils. Mathematics achievement was very high by American standards. (Easley, 1984, on a Japanese project)

We need to recognise the importance of verbalisation. Putting thoughts into words requires students to organise their thinking and to confront their incomplete understanding. Listening to others affords them the opportunities to contemplate the thinking of others and to consider the implications for their own understanding. (California State Department of Education, 1987)

This article is based on the first year of a longitudinal study intended to shed light on the question of whether or not discussion in the mathematics classroom is an aid to mathematical understanding. Before looking further at the research context in which this project is set, we wish to offer our definitions of the two notions in the title.

MATHEMATICAL DISCUSSION

"Discussion between teacher and pupils and pupils themselves" has been claimed to be one of the six elements which should be included in mathematics teaching at all levels (Cockcroft, 1982). Although this statement is widely cited and approved it remains the case that few have observed genuine discussion in mathematics classrooms let alone conducted research into its effectiveness. For example, the Classroom Language Observation Group led by Shuard and Kerslake collected transcripts of
mathematics lessons which claimed to embody discussion but almost all turned out, on detailed analysis, to be heavily didactic, teacher-led adjuncts to exposition (Shuard, 1984).

The Primary Survey of England and Wales criticised current practice saying:

... more attention could usefully have been given to more precise and unambiguous use of ordinary language to describe the properties of number, size, shape or position. (DES, 1978)

Similarly, the Secondary Survey Report claimed that:

... the potential of mathematics for developing precision and sensitivity in the use of language was underused. (DES, 1979)

These criticisms, however, appear not to be so much a case for teacher-pupil or pupil-pupil discussion as a case for clear teaching which emphasises the meaning of mathematical statements and develops the skills of oral communication, not necessarily solely for mathematical purposes. It is not our intention to dispute the value of discussion as a general learning technique, nor question the need for careful development of language. It is certainly true that the ability to make precise and unambiguous use of ordinary language could be developed by talk about mathematics which is structured towards this end. Such use of language could also be assisted by the experience of being forced to negotiate common meanings in the course of a group discussion. In practice, however, there is a tendency for classroom talk to be extremely imprecise; the very phrase "precise and unambiguous use of ordinary language" (our emphasis) is close to being contradictory.

A wide variety of talk may occur within a mathematics lesson, and we acknowledge that much of it may be valuable. For the purpose of the present project, however, we intend to focus only on 'mathematical discussion'. Before offering a precise definition of what we mean by this, we mention two examples of talk which are excluded from our sphere of interest. Teacher-led talk which is merely an adjunct to exposition is not genuine discussion because typically it consists not of pupils formulating their own opinions but of pupils guessing the correct answers required to satisfy questions posed by the teacher. Similarly there is much pupil talk which, although it does consist of the pupils' own opinions, cannot qualify as mathematical discussion because it is not mathematical or because it is not interactive. This is not to say that one person talking at another might not gain in understanding but that it is not a discussion. We therefore adopt the following definition of mathematical discussion:
It is purposeful talk
i.e., there are well-defined goals even if not every participant is aware of them. These goals may have been set by the group or by the teacher but they are, implicitly or explicitly, accepted by the group as a whole.

on a mathematical subject
i.e., either the goals themselves, or a subsidiary goal which emerges during the course of the talking, are expressed in terms of mathematical content or process.

in which there are genuine pupil contributions
i.e., input from at least some of the pupils which assists the talk or thinking to move forwards. We are attempting here to distinguish between the introduction of new elements to the discussion and mere passive response, such as factual answers, to teachers' questions.

and interaction
i.e., indications that the movement within the talk has been picked up by other participants. This may be evidenced by changes of attitude within the group, by linguistic clues of mental acknowledgement, or by physical reactions which show that critical listening has taken place, but not by mere instrumental reaction to being told what to do by the teacher or by another pupil.

MATHEMATICAL UNDERSTANDING
Having explained what we mean by ‘discussion’ it is necessary to be explicit about our view of ‘understanding’ since the word is used by different authors in different senses (Schroeder, 1987). We consider understanding to encompass the comprehension of concepts, the relationships between these concepts and ordinary language or physical objects. Such comprehension must also include the procedural and process skills which depend upon familiarity with these relationships. It is also upon these relationships that understandings of ‘why a piece of mathematics works’, as distinct from ‘how to do it’ will be based. Deep mathematical understanding must therefore be primarily relational understanding (Skemp, 1976). However, the nature of mathematics, whereby procedures at one level can, as it were, be put ‘on automatic’ so as to free conscious thought for concepts at a higher level, means that instrumental or procedural understanding is an essential part of the totality of mathematical understanding and will also be observed when relational understanding exists. It is not always easy to
distinguish such instrumental or procedural understanding from the effects of rote learning which leads to the ability to perform a particular task with no idea, either currently or in the past, of why the task is performed in this way. We do not regard such ability as demonstrating mathematical understanding but neither do we regard apparently instrumental functioning as evidence of any lack of deeper understanding (Pirie, 1988).

In the same way we do not regard as evidence of lack of mathematical understanding what is merely a mismatch of language or notation. The pupil will inevitably construct personal conceptual frameworks, albeit based partially on the visible understanding and contexts presented by the teacher, and these may show linguistic or notational differences when compared with an 'official' version. For example, consider the often quoted example of addition of fractions (Hart, 1981; Kerslake, 1986). Some pupils may successfully write $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ on the basis of rote learning and be shown in an interview to have very little mathematical understanding. At the other extreme a pupil whose personal alternative framework is to write $\frac{b}{c}$ meaning "b out of c", as in "b out of c test questions were answered correctly", and to write + meaning "and", will arrive at the most commonly observed 'mistake' $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$ using mathematical reasoning based on understanding (Confrey, 1987).

From this standpoint, therefore, mathematical understanding is not something which can be tested by any simple checklist of questions. Its presence may be inferred but not measured directly. Important elements leading to such inference may be a display of awareness of relationships between concepts, of ability to adopt procedures when faced with variations in the data or language or context provided, of possession of good generic examples of abstract concepts used, or of fluency with symbolism. There will be times when indications of the degree of mathematical understanding becomes evident only as a result of a sensitive interview. There will be other times when the presence, or lack, of mathematical understanding will be displayed so obviously as to need no further analysis.

THE CONTEXT OF THE PROJECT

One project which was deliberately structured to produce classroom discussion was that conducted by Easley at the University of Illinois. He had observed teaching at certain primary schools in Japan where the class was divided into small groups of children each with their own temporary 'leader' who was responsible for mathematical discussion within the group, and who then presented the group's ideas to the whole class.
In Kitamaeno School, use of peer group dialogue helped children recognise alternative schemes and deal with them. Every child in the classroom would be involved in the thinking process, even with classes over forty. (Easley, 1984)

We are not aware of any similar systematic attempt to produce mathematical discussion in secondary schools. There are many factors which prevent the assumption that observed benefits of discussion at the primary school level will necessarily also be present at the secondary school: the different nature of pupils' expectations, the differences in school structure, the effects of adolescence, the possibility that ideas about certain mathematical topics became 'set' at some point in the primary years, and the effect of external examinations all need to be considered.

Cognisance must also be taken of relevant research into linguistics in the mathematics classroom. The review by Austin and Howson (1979) makes the helpful distinction between the language of the learner, the language of the teacher and the language of mathematics; this distinction is essential to any study of the role of language in the development of mathematical concepts.

Hoyles (1985) makes the additional important distinction between the cognitive function of talk and the communicative function of talk. The first includes both the period of silence during which a participant attempts to incorporate another's framework into his own framework and talk which enables the speaker to step back and reflect upon a mathematical idea. The second includes communicating an idea effectively to another, or challenging and logically rejecting another's point of view. A more problematic aspect of her approach is the suggested definition of mathematical understanding as the ability to act in all these ways. Our own perspective is that such actions are evidence of mathematical understanding but not part of understanding itself.

One project, explicitly incorporating small group discussion, was the Diagnostic Teaching Project (A. W. Bell et al., 1986). Here the initial aim of the teaching method was to draw out pupils' misconceptions. This has two advantages: a misconception, as distinct from a regurgitated 'right answer', is more likely to be the pupil's own, and a mistake yields more information about what the pupil is thinking than does a correct answer (Schwarzenberger, 1984). The project then proposed a teaching sequence designed to eliminate the misconceptions. The misconceptions found were used to set up conflict between alternative cognitive frameworks to be resolved in small group discussion. Researchers attempted to rate the quality of the discussion and of the progress made in resolving the conflict. This aspect of the project, however, does not appear to yield convincing
evidence, although the intention is attractive. Subsequent work by G. H. Bell (1987), based on pupils' work with SMP 11–16 booklets, found that the addition of small group discussion in this case did not necessarily improve learning and that good discussion did not automatically result when conflict situations were presented.

**DESIGN OF THE STUDY**

It is necessary to be wary of the gap between how we would idealistically like children to learn and the realities of the classroom. For this reason the design of the first year of our own study was based on observation of four particular mathematics classrooms as they exist, with no attempt to impose or stimulate any special activity.

The intention was to study the classes of teachers who used discussion as an intentional component of their teaching styles. Such teachers were in fact hard to find, and for this reason their classes must be considered atypical. We were fortunate, in that the schools we used consisted of a small selective rural school, a large comprehensive school in a predominantly affluent area, an inner-city school with a large majority of pupils from first-generation immigrant families, and a comprehensive school serving a mixed catchment area of industrial, farming and mining communities. This variety ensured that our results would at least not be biased towards any particular type of school or social mix of population. The sample was however deliberately weighted in favour of finding examples of mathematical discussion since the teachers had been recommended to us as encouraging such activity and also knew that we were interested in it.

In each class a small group of children was chosen as the site of our tape-recorder. They were told that they were not being tested in any way, the tapes would not be played to their teachers and that we were merely interested in how pupils talk about mathematics.

Collaborative groupwork has been extensively researched – Yeomans (1983) offers a good review of this field – but when relating our project to other research it is necessary to distinguish between apparent groupwork, where pupils are sitting in small groups but in effect working independently, and groupwork with the collaboration and interaction which would be a prerequisite for genuine discussion (Bennett, 1978; Harwood, 1988). Some observations of groupwork, especially at the primary level, have involved mathematical tasks. However, the review by Bennett (1985) points out that the results obtained are often based upon group tasks imposed by the researcher, as distinct from groupwork arising from the teacher's usual
classroom practice, and are often more concerned to measure the apparent group achievement (e.g., in solving a problem) as distinct from study of the interactions within a group.

The small groups of pupils on which we focused were initially interviewed to discover their attitudes to mathematics and to different styles of mathematics teaching. Some time after the observed lessons, we again interviewed the pupils together as a group, in an attempt to elicit their understanding of the topics on which they had been working. We used a loosely structured clinical interviewing technique (Ginsburg, 1981, 1983) in which the pupils talked their way through a task and the interviewer followed the paths of thinking which they evidenced, sometimes returning later to probe more deeply into relevant key concepts (Pirie, 1988).

**OCCURRENCE OF DISCUSSION**

Analysis of classroom tapes shows that the instances of talk which fitted our definition of discussion are comparatively rare. The most common reason was that there was no interaction between the participants: pupils would talk through a piece of mathematics aloud but with no evidence of reaction by a listener. A less common reason was that, although interactive, the goals were not well-defined: the talk had a scatter-shot nature going in no particular direction. Note that we are not denying such brainstorming may be a good thing; we are merely observing that it is not discussion.

Interestingly, fragments of social chat did not in themselves prevent or inhibit mathematical discussion. Such fragments occur throughout our tape recordings, both in episodes which satisfy our definition of mathematical discussion and in episodes which do not. We conjecture that such light relief may actually be necessary for most pupils to sustain discussion on a mathematical topic.

**ANALYSIS OF DISCUSSION**

While ability to talk purposefully about mathematics is not synonymous with mathematical understanding, it is clearly important *prima facie* evidence. In a similar way, ability to write mathematics, or to perform practical work, or to undertake a successful investigation, are all possible indications of the existence of mathematical understanding. In all these cases, supplementary corroborative evidence might come from the context of the activity or from subsequent interviews with the pupils.

Both in the interviews and in the analysis of the classroom transcripts, we
found it necessary to distinguish between two very different kinds of statement: reflective statements, which describe concepts and the relationships between them and are closely linked to relational understanding, as against operational statements which describe actions and are closely linked to instrumental understanding.

Although it may indeed be an important skill to state the meaning of a mathematical statement in a non-mathematical language, especially if it is hoped to link the mathematical theory to concrete contexts from which the mathematics might have arisen, it must not be forgotten that the power of mathematics also derives from the ability to temporarily forget such meanings while working within the mathematical symbolism.

We need, therefore, to distinguish further between the language being used and the statements being made. Reflective statements can be made in ordinary language and conversely new mathematical language can enable operational statements to be made in new and powerful ways. A good historical example of this is the introduction by Leibniz of the notation \( dx \).

This distinction is important because it is tempting to assume that pupils who use ordinary language are not displaying relational understanding. In reality no such assumptions can be made without examining the pupils' statements closely, if possible together with observation of the accompanying actions. It is possible to analyse the local structure of classroom mathematical talk by some form of discourse analysis (Coulthard and Sinclair, 1975). This local structure - type of question, type of response, new statement - can be revealing when applied to teacher-pupil exchanges but we have found it less useful in analysing talk between one pupil and another.

We are therefore led to classify episodes of discussion in terms of three parameters describing the focus of the discussion, the kind of language used and the type of statements being made. Each episode is encoded according to the following scheme.

**First parameter:** What is it that gives the speakers something to talk about?

a. They have a task or concrete object as the focus of their talk.
b. They do not have an understanding of something but they know this and it gives them something to talk about.
c. They have some understanding and this gives them something to talk about.
Second parameter: What level of language is being used?

f. They lack appropriate language; they do not have the right or useful words.
g. They use ordinary language.
h. They use mathematical language.

Third parameter: What kind of statements are being made?

p. Incoherent statements; that is, incoherent to the other participants.
q. Operational statements about what to do or how to do it.
r. Reflective statements offering explanations or attempts to move beyond the immediate task.

It must be stressed that we have not been concerned with the analysis of single utterances, but with that of episodes of discussion.

We offer here two episodes to illustrate this method of categorisation. The first, a straightforward episode, involves four 12-year-old girls who are tackling an investigation ("Frogs") in which black and white cubes are moved by slides and jumps.

Ann-Marie: I did it, I should have remembered it.
Joanne: You can't remember can you.
Tracey: You're supposed to remember them things but she moved them too quick.
Susanne: I know how it works - you have the whites on one side.
Tracey: O.K. You have to have the whites a certain side don't you.
Ann-Marie: I know!
Joanne: How do you reckon we are going to record this?
Susanne: So that we can remember it.
Tracey: I'll count how many moves you make.
Joanne: Come on – watch her.
Ann-Marie: Can't I jump that one? Can't I do that?
Joanne: Do it again and I'll count how many moves you make.

The task is to record their solution in some way, and they decide to record the number of moves made, by counting. They use ordinary language and make operational statements. The episode is therefore encoded (a, g, q).

The second episode is more complex, and involves three 12-year-old-girls. Having done several examples using numerically specified lengths,
the class has been asked to find the area of a trapezium with parallel sides, length $j$ and $k$, and height $h$.

Katie: So hang on, it's ... $j$ plus $k$ (writing)
Alison: Divided by 2
Katie: Divided by 2 (writing)
Alison: No, times $h$
Katie: Times $h$ ... equals
Alison: Equals what?
Carol: $j$ plus $k$
Katie: Divided by 2 times $h$
Carol: But you have to know what $j$ and $k$ are before you can . . . .
Katie: No, you've got to have an answer with $j$ and $k$ and . . . .
Katie: You've got to divide by 2. You divide by 2 always so it's $j$ plus $k$.

Alison in unison
Carol
Katie

And then what does it equal?

Katie: $j$ plus $k$ over 2 times $h$

(laughter)
Carol: What did you do?
Alison: You've not got to do the numbers.
Carol: Do you have to do the answer?
Katie: Your answer is that, isn't it?
Katie: Look, you answer that . . . equals . . . .
Carol: That's what the answer is . . . that
Katie: Yeah.
Carol: It's got to be.
Katie: The answer is $j$ plus $k$ over 2 times $h$.

The language used is mathematical. Note that this categorisation may depend on the maturity of the participants: what is mathematical for young pupils may be considered ordinary language by older pupils. The episode begins with operational statements focused on the task set but soon moves into reflective statements focused on the pupils' lack of understanding. The episode moves from being encoded $(a, h, q)$ to being encoded $(b, h, r)$, as they resolve the dilemma raised by their initial desire to complete the equation $\frac{j+k}{2} \times h =$
SUMMARY

This paper has set out the background to our project and outlines a method of classification which we expect will prove useful in investigating links between mathematical discussion and mathematical understanding. This framework for analysis has already allowed us to ask questions about the nature of any such relationships, for example, "To what extent does lack of mathematical language inhibit reflective statements?", "Is it possible to identify what causes pupils to shift from one kind of discussion to another?", "What are the characteristics of discussion stimulated by shared awareness of lack of understanding?" The crucial issue of how we are attempting to probe understanding through clinical interviews is dealt with in more detail elsewhere (Pirie, 1988). As stated earlier, we are reporting on the initial stage of a longitudinal study focussing in depth on a small group of pupils in each of four different classrooms. We hope that our observations and analysis will lead to recommendations for future research, but at this stage we can neither substantiate nor deny the attractive hypothesis of a causal relationship between mathematical discussion and mathematical understanding.

REFERENCES

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